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Comparison between classical and Bayesian estimation with joint Jeffrey's prior to Weibull distribution parameters in the presence of large sample conditions

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Abstract

Weibull distribution has been considered one of the most common and valuable distributions for building and analyzing good models for lifetime data. Many researchers have studied the properties of Weibull distribution, also in search of the best method to estimate both parameters. In this paper, we proposed a comparison of Weibull distribution parameters under large sample conditions. We chose to study the classical estimation methods of Weibull distribution parameters, including the maximum likelihood estimator and moments estimation (ME). Next, we compared these methods with the Bayesian estimation method (BE) with Jeffrey's prior function. We validated the proposed study via simulation using both small and large samples. We used mean square errors (MSE) to determine the best estimation method. Our simulation findings suggest that maximum likelihood estimators are reasonably effective when using small sample sizes. In addition, in cases where the sample size is larger, the BE performed more effectively for both scale and shape parameters of the Weibull distribution function.

Key words: Weibull distribution, classic estimation, Bayesian estimation, Jeffrey's prior, large sample.

1. Introduction

Weibull distribution was first introduced by Waloddi Weibull (1951), and it has been widely used in reliability and life data analysis. Also, the Weibull distribution function can be used as a model for various life behaviors depending on the values of its parameters. Estimation of parameters for the Weibull distribution function is fundamental. There are two parameters of the Weibull distribution function; the first is the Shape Parameter β, which marks the behavior of the distribution. Different values of β give the Weibull distribution function variety.

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Moreover, the Shape Parameter affects the failure rate of the distribution function in life data analysis. The second Weibull distribution function is the Scale Parameter α, which determines the probability density function's figure and peak. The height of the probability density function will decrease as α increases.

Many approaches have been submitted to estimate the two parameters of the Weibull distribution function, and many were considered classical methods of estimation, such as the Maximum Likelihood Estimator that depends on finding the values of parameters that maximize the joint probability function of observed data over the parameter space. In addition, the Moments Estimator starts by expressing the population as a function of the parameters. It is set to be equal to the sample moments to get equations and solve them by finding an estimator for the parameters. Other modern estimation methods were submitted, such as the Bayes Estimation method, which depends on knowledge about the prior distribution of the parameters to develop new and efficient estimators using Bayes Theory. Bayes estimation depends on selecting recently developed functions, such as Jeffrey's prior and Gamma-Gamma prior functions.

Over the years, many prior functions were submitted to be Informative and Non-Informative prior functions to get good quality estimators for Weibull distribution function parameters. All estimation methods agree on minimizing the difference between the observed value and the fitted value provided by the distribution function.

This paper is organized as follows: Section 2 discusses related work, Section 3 presents the Maximum Likelihood Estimation, Section 4 describes the Moment Estimator, Section 5 presents the Bayes Estimator, Section 6 presents comparison methods, Section 7 presents the discussion and results, and Section 8 presents conclusions.

2. Related Work

Weibull distribution has been widely studied by many researchers, such as:

(Mann, Schafer, & Singpurwalla, 1974) proposed a study of the analysis of reliability and life data that used the Weibull distribution function. They estimated the parameters in many graphical and analytical methods then studied the effect of parameter estimation methods in the reliability function.

(Popocikova & Sedliackova, 2014) compared different estimators for the Weibull distribution function for both shape and scale parameters. They studied the parameters' performance using the Weighted Least Square (WLS), Maximum Likelihood Estimator, and Moments Estimation methods (ME). The comparisons were based on Monte Carlo Simulation data using Minimum Square Errors as a comparison tool.

The following researchers discussed the excellent qualities of the Bayesian estimator using different functions:

(Aslam, Kazmi, Ahmad, & Shah, 2014) estimated the Weibull distribution function's shape and scale parameters using the Bayes Estimation method. They considered Informative and Non-Informative prior functions for both parameters. They also used many loss functions to improve the estimation. A comprehensive simulation was used to make a fair comparison among different Bayes estimates.

(Guure, Ibrahim, & Ahmed, 2012) proposed a study to estimate the Weibull distribution function parameters using classical Maximum Likelihood Estimation, Moments Estimation (ME), and Bayes Estimation method. They used Jeffrey's prior function as a Non-Informative prior function, and three types of loss functions to improve the Bayes estimates. Also, Minimum Square Errors were used to compare estimates of shape and scale parameters.

In this paper, we estimated the parameters of the Weibull distribution under the presence of a large sample size under study. We studied the performance of three estimation methods by changing the shape parameter under different sample sizes.

Estimating Weibull distribution parameters is a significant process used in different areas, including reliability and modeling lifetime data for engineering and medicine applications.

3. Maximum Likelihood Estimator

One of the most commonly used methods to estimate the parameter of any known distribution function is the Maximum Likelihood Estimation method, which utilizes the log-likelihood function to estimate parameters.

Let $x_1, x_2, \dots, x_n$ be a random sample with *n* the Weibull Distribution pdf, which will be given as (Guure, Ibrahim, & Ahmed, 2012).

$$f(x) = \left(\frac{\beta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\beta-1} exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right]. \tag{1}$$

where (α, β) are the scale and shape parameters as $\alpha, \beta > 0$. Then, the likelihood of the pdf in (1) $f(x_1, x_2, \dots x_n \backslash \alpha, \beta)$ will be (Nwobi & Ugomma, 2014).

$$L(x_1, x_2, \dots x_n \backslash \alpha, \beta) = \prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{\beta-1} exp\left[-\left(\frac{x_i}{\alpha}\right)^{\beta}\right]\right\}. \tag{2}$$

The log-likelihood function is (Johnson, Kotz, & Balakrishnan, 1994), (Kundu, 2008).

$$\ln(L) = n\ln(\beta) - n\beta\ln(\alpha) + (\beta-1)\sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} \tag{3}$$

By differentiating (3) respectively to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and equating to zero we get

$$\frac{\partial \ln L}{\partial \alpha} = -n\left(\frac{\beta}{\alpha}\right) + \left(\frac{\beta}{\alpha}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta} = 0. \tag{4}$$

$$\frac{\partial \ln L}{\partial \beta} = \left(\frac{n}{\beta}\right) + \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\ln\left(\frac{x_i}{\alpha}\right) = 0 \tag{5}$$

From (4) we can obtain (Zhang & Meeker, 2005)

$$\hat{\alpha} = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}\right]^{1/\beta} \tag{6}$$

It is useful here to use numerical methods to find the value of $\hat{\boldsymbol{\beta}}$. If we consider that $\boldsymbol{f}(\boldsymbol{\beta})$ is the same in (5) we can use the Newton-Raphson method by taking the first differential of $\boldsymbol{f}(\boldsymbol{\beta})$ as below (Lawless, 2011).

$$f'(\beta) = -\left(\frac{n}{\beta^2}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\ln^2\left(\frac{x_i}{\alpha}\right) \tag{7}$$

Substituting (6) in (5).

$$f(\beta) = \left(\frac{n}{\beta}\right) + \sum_{i=1}^{n}\left[\frac{(x_i)}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}\right]^{1/\beta}}\right] - \sum_{i=1}^{n}\left[\frac{(x_i)^{\beta}}{\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}}\right]\ln\left[\frac{(x_i)}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}\right]^{1/\beta}}\right] \tag{8}$$

Substituting (6) in (7).

$$f'(\beta) = -\left\{\left(\frac{n}{\beta^2}\right) + \sum_{i=1}^{n}\left[\frac{(x_i)^{\beta}}{\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}}\ln^2\frac{x_i}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}\right]^{1/\beta}}\right]\right\}. \tag{9}$$

Then, by choosing an initial value for $\boldsymbol{\beta_i}$ we can obtain $\hat{\boldsymbol{\beta}}$ by iterating the formula below until it converges to the MLE for $\boldsymbol{\beta}$.

$$\beta_{i+1} = \beta_i - \frac{\left(\frac{n}{\beta}\right) + \sum_{i=1}^{n}\left[\frac{(x_i)}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}\right]^{1/\beta}}\right] - \sum_{i=1}^{n}\left[\frac{(x_i)^{\beta}}{\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}}\right]\ln\left[\frac{(x_i)}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}\right]^{1/\beta}}\right]}{-\left\{\left(\frac{n}{\beta^2}\right) + \sum_{i=1}^{n}\left[\frac{(x_i)^{\beta}}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}\right]}\ln^2\frac{x_i}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i)^{\beta}\right]^{1/\beta}}\right]\right\}}. \tag{10}$$

4. Moments Estimator

The method of moments is commonly used depending on obtaining the K_{th} Moment *Mk* of the Weibull distribution function and equating it with the sample K_{th} moments given by (Pobočıková & Sedliačková, 2014).

$$\boldsymbol{M_k} = \frac{1}{n}\sum_{i=1}^{n} x_i^k \tag{11}$$

Then, $\boldsymbol{M_1} = \bar{x} = E(x)$, which is equal to the expected value of the Weibull distribution function.

$$\beta\Gamma\left(1+\frac{1}{\alpha}\right) = \bar{x} \tag{12}$$

$$\beta^2\Gamma\left(1+\frac{2}{\alpha}\right) = \frac{1}{n}\sum_{i=1}^{n} x_1^2 \tag{13}$$

And by dividing (13) on the square of (12)

$$\frac{\Gamma\left(1+\frac{2}{\alpha}\right)}{\Gamma^2\left(1+\frac{1}{\alpha}\right)} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_1^2}{\bar{x}^2} \tag{14}$$

There is no analytical solution for (14), so we can use numerical methods to estimate $\boldsymbol{\alpha}$. Ramerez and Carta (2005) gave a starting point for $\boldsymbol{\alpha}$ such that:

$$\hat{\alpha} = \left(\frac{\bar{x}}{S_x}\right)^{1.086} \tag{15}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ then from (12).

$$\hat{\beta} = \frac{\bar{x}}{\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)} \tag{16}$$

5. Bayes Estimator

The Bayesian estimation method is critical and has attracted much attention lately. The Bayesian approach begins with determining a prior distribution function for the parameters under study. When we have information about the parameters, we may use the Informative prior distribution function or the Non-Informative prior distribution function. We have no knowledge about the parameters here, so we used the Non-Informative prior function. The most common prior distribution function is Jeffrey's prior. Jeffrey (Guure, Ibrahim, & Ahmed, 2012) suggested using the square root of the determinant of the Fisher information matrix as a prior distribution function for the parameters such that $u(\alpha,\beta) = \sqrt{\det(I_{(\alpha,\beta)})}$ where:

$$I_{(\alpha,\beta)} = \begin{bmatrix} E\left(\frac{\partial^2 \log(f_{(x)})}{\partial^2\alpha^2}\right) & E\left(\frac{\partial^2 \log(f_{(x)})}{\partial\alpha\partial\beta}\right) \\ E\left(\frac{\partial^2 \log(f_{(x)})}{\partial\alpha\partial\beta}\right) & E\left(\frac{\partial^2 \log(f_{(x)})}{\partial^2\beta^2}\right) \end{bmatrix} \tag{17}$$

According to (Guure, Ibrahim, & Ahmed, 2012), the final result for the prior distribution function will be as follows:

$$u(\alpha,\beta)=\frac{1}{\alpha\beta} \tag{18}$$

Since we have satisfied the prior distribution function, we can now compute the posterior distribution function according to Bayes theory, in which the joint density function of α,β is:

$$f(\alpha,\beta\backslash x_1,x_2,\dots .x_n)=\frac{f(x_1,x_2,\dots x_n\backslash\alpha,\beta)u(\alpha,\beta)}{\int_0^\infty\int_0^\infty f(x_1,x_2,\dots x_n\backslash\alpha,\beta)u(\alpha,\beta)d\beta d\alpha} \tag{19}$$

By using the likelihood function:

$$f(\alpha,\beta\backslash x_1,x_2,\dots .x_n)=\frac{L(x_1,x_2,\dots x_n\backslash\alpha,\beta)u(\alpha,\beta)}{\int_0^\infty\int_0^\infty L(x_1,x_2,\dots x_n\backslash\alpha,\beta)u(\alpha,\beta)d\beta d\alpha} \tag{20}$$

Then:

$$f(\alpha,\beta\backslash x_1,x_2,\dots .x_n)=\frac{\frac{1}{\alpha\beta}\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{\beta-1}exp\left[-\left(\frac{x_i}{\alpha}\right)^{\beta}\right]\right\}}{\int_0^\infty\int_0^\infty\frac{1}{\alpha\beta}\prod_{i=1}^{n}\left\{\left(\frac{\beta}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{\beta-1}exp\left[-\left(\frac{x_i}{\alpha}\right)^{\beta}\right]\right\}d\beta d\alpha}. \tag{21}$$

Therefore, the Bayes estimators for the parameters will be:

$$BE_\alpha=E(\alpha\backslash x_1,x_2,\dots .x_n)=\int_0^\infty\alpha f(\alpha,\beta\backslash x_1,x_2,\dots .x_n)d\alpha \tag{22}$$

$$BE_\beta=E(\beta\backslash x_1,x_2,\dots .x_n)=\int_0^\infty\beta f(\alpha,\beta\backslash x_1,x_2,\dots .x_n)d\beta \tag{23}$$

In addition, we suppose that α , β are independent.

6. Comparison Method

Different statistical tools can be used to make a fair comparison among estimators, and here we selected the MSE Mean Squared Errors, which is given by.

$$MSE=\frac{1}{n}\sum_{i=1}^{n}\left[\hat{F}(x_i)-F(x_i)\right]^2 \tag{24}$$

where $F(x_i)$ is the cumulative distribution function for the Weibull distribution as follows (Pobočıková & Sedliačková, 2014):

$$F(x_i)=\begin{cases}1-exp\left\{-\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}, & x\geq 0\\ 0, & otherwise\end{cases} \tag{25}$$

Thus, we can use parameters and their estimators to substitute in (25).

7. Simulation Study

In this section, we used a MATLAB program to make a Monte Carlo simulation to generate samples of random variables with the Weibull distribution; a Monte Carlo simulation method depends on generating initial random variables with Uniform Distribution and then generating the Weibull Distribution according to its cumulative distribution function. We can summarize the Monte Carlo Simulation for Weibull Distribution in the following steps:

1. Generating Normal z by using Lehmer's recursion simple random generator which is $\boldsymbol{z = a z_0 \ mod \ m}$ where $\boldsymbol{z_0 = 1}$, $\boldsymbol{a = 3}$, $\boldsymbol{m = 31}$ where a and m can be changed to have a cycle of random numbers.
2. Normalizing z to obtain a random variable (u) with a value between zero and one $\boldsymbol{u = z/m}$
3. Obtaining t from the cumulative distribution function of the Weibull Distribution by equalizing it to u, i.e. $\boldsymbol{u = 1 - e^{-\left(\frac{t}{\alpha}\right)^{\beta}}}$ then $= \boldsymbol{\alpha \sqrt[\beta]{-\ln u}}$.

Then, t is a random variable with the Weibull Distribution, and we repeat these steps for 300 times in order to get random samples. Here, we chose only three values for shape parameter $\boldsymbol{\beta = 1.5, 2, 2.5}$ and we chose one value for scale parameter $\boldsymbol{\alpha = 0.5}$. These selections for the parameters' values were used to generate random samples [9], and here we chose sample sizes $\boldsymbol{n = 10, 20, \ldots 120}$ to cover both small and big samples.

Table 1: The estimated values of $\hat{\alpha}$, $\hat{\beta}$ and the obtained MSE with three estimation methods: Maximum Likelihood Estimation Method, Moment Estimation and Bayes Estimation, when $\alpha = 0.5$, $\beta = 1.5$

n	MLE			ME			BE		
	$\hat{\alpha}$	$\hat{\beta}$	MSE	$\hat{\alpha}$	$\hat{\beta}$	MSE	$\hat{\alpha}$	$\hat{\beta}$	MSE
10	0.7821	2.2763	2.7820	0.9013	2.7011	3.3212	0.9480	2.3631	2.8821
20	0.7612	2.2283	2.7723	0.8877	2.6699	3.2772	0.8931	2.3231	2.8271
30	0.7537	2.1843	2.7423	0.8721	2.6278	3.2242	0.8430	2.3141	2.7741
40	0.7411	2.1463	2.7024	0.8600	2.5790	3.1811	0.7921	2.2451	2.7231
50	0.7201	2.1133	2.6623	0.8511	2.5401	3.1404	0.7420	2.2071	2.6741
60	0.7015	2.0856	2.6128	0.8489	2.5154	3.1002	0.6961	2.1711	2.6291
70	0.6714	2.0456	2.5523	0.8401	2.4853	3.0598	0.6622	2.1276	2.5731
80	0.6421	2.0421	2.4732	0.8362	2.4644	3.0209	0.6180	2.0926	2.4921
90	0.6001	2.0412	2.4123	0.8302	2.4501	2.9849	0.6000	2.0576	2.4111
100	0.5598	2.0213	2.3576	0.8277	2.4400	2.9509	0.5381	2.0221	2.3301
120	0.5342	1.9963	2.3041	0.8223	2.4306	2.9199	0.5221	1.9891	2.2491
140	0.5210	1.9743	2.2598	0.8204	2.4214	2.8791	0.5091	1.9561	2.1701
160	0.5189	1.9633	2.2087	0.8188	2.4123	2.8341	0.5042	1.9251	2.0951
180	0.5045	1.9243	2.1665	0.8161	2.4037	2.8021	0.5011	1.8932	2.0241
200	0.5001	1.9153	2.0912	0.8155	2.3944	2.7711	0.5000	1.6551	1.9541

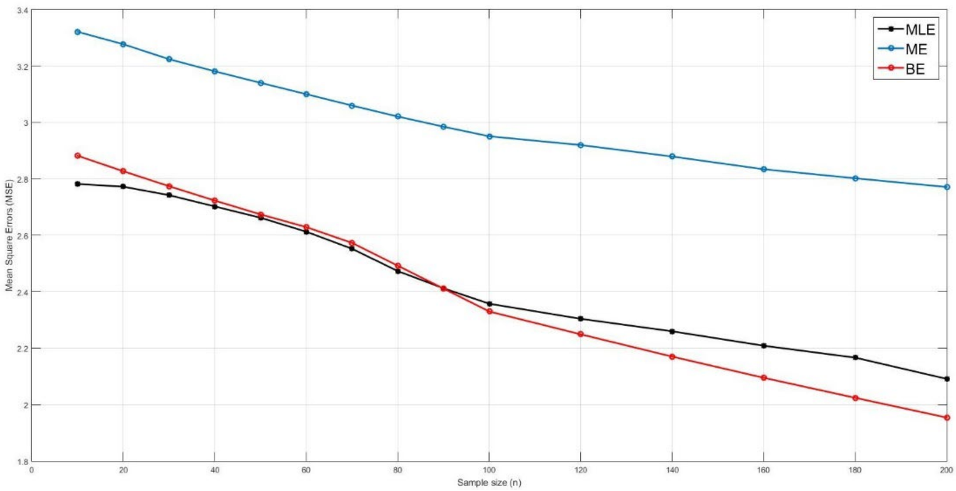

Figure 1: The values of MSE for the three estimation methods: Maximum Likelihood Estimation Method, Moment Estimation, and Bayes Estimation, when $\alpha = 0.5$, $\beta = 1$

While we have $\boldsymbol{\beta = 1.5}$, $\boldsymbol{\alpha = 0.5}$, Table 1 and Figure 1 show the effectiveness of the MLE estimator with a small sample size, and this continues until we get to sample sizes of 90 as shown in Figure 1. The BE estimator showed great performance and improved as the sample size increased. The ME estimator showed no priority with both small and large sample sizes.

Table 2: The estimated values of $\hat{\alpha}$, $\hat{\beta}$ and the obtained MSE with three estimation methods: Maximum Likelihood Estimation Method**,** Moment Estimation and Bayes Estimation when $\alpha = 0.5$, $\beta = 2$

n	MLE			ME			BE		
	α^	β^	MSE	$\hat{\alpha}$	$\hat{\beta}$	MSE	$\hat{\alpha}$	$\hat{\beta}$	MSE
10	0.8371	2.6216	3.7820	0.9013	2.9011	4.3321	0.8780	2.7031	3.8755
20	0.8161	2.6006	3.7611	0.8801	2.8801	4.2121	0.8510	2.6721	3.8445
30	0.7951	2.5796	3.7411	0.8601	2.8601	4.1521	0.8351	2.6421	3.8145
40	0.7761	2.5606	3.7221	0.8411	2.8411	4.0931	0.8001	2.6131	3.7855
50	0.7581	2.5426	3.7041	0.8231	2.8231	4.0751	0.7761	2.5851	3.7575
60	0.7411	2.5256	3.6871	0.8016	2.8061	4.0571	0.7531	2.5581	3.7305
70	0.7251	2.5096	3.6721	0.7901	2.7901	4.0411	0.7311	2.5331	3.7045
80	0.7101	2.4936	3.6581	0.7751	2.7751	4.0261	0.7101	2.5091	3.6785
90	0.6961	2.4786	3.6451	0.7611	2.7611	4.0121	0.6901	2.4861	3.6525
100	0.6831	2.4684	3.6341	0.7481	2.7481	3.9991	0.6711	2.4641	3.6275
120	0.6731	2.4469	3.6231	0.7361	2.7361	3.9871	0.6531	2.4441	3.6035
140	0.6581	2.4366	3.6131	0.7251	2.7251	3.9761	0.6361	2.4241	3.5805
160	0.6461	2.4236	3.6041	0.7151	2.7151	3.9661	0.6201	2.4051	3.5585
180	0.6361	2.4116	3.5951	0.7061	2.7061	3.9571	0.6041	2.3671	3.5375
200	0.6231	2.4006	3.5871	0.6981	2.6981	3.9491	0.5901	2.3701	3.5165

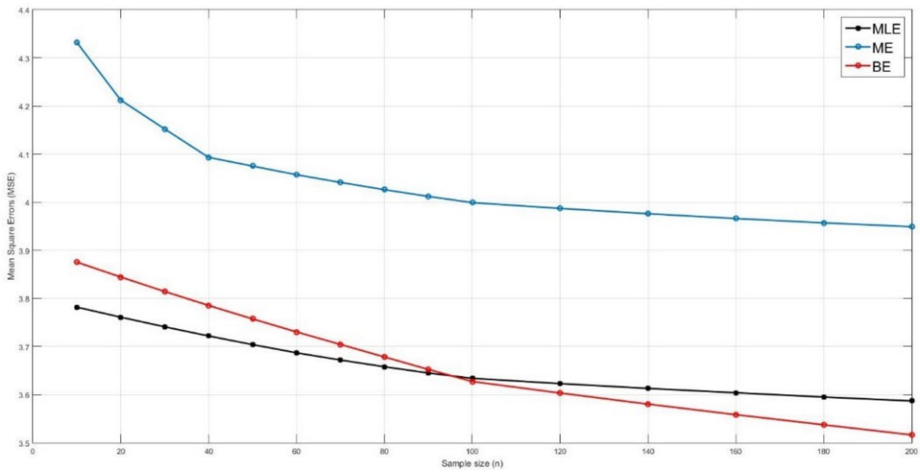

Figure 2: The value of MSE for the three estimation methods: Maximum Likelihood Estimation Method, Moment Estimation, and Bayes Estimation, when $\boldsymbol{\alpha = 0.5}$, $\boldsymbol{\beta = 2}$

By changing $\boldsymbol{\beta = 2}$ and from Table 2 and Figure 2, we see similar results, but here the MLE estimator's performance is good only to sample sizes of 100. The BE estimator becomes a better estimator until sample sizes reach 200, where the ME estimator's performance is not good as the sample size changes.

Table 3: The estimated values of $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$ and the obtained MSE with three estimation methods: Maximum Likelihood Estimation Method, Moment Estimation and Bayes Estimation $\boldsymbol{\alpha = 0.5}$, $\boldsymbol{\beta = 2.5}$

n	MLE			ME			BE		
	$\hat{\alpha}$	$\hat{\beta}$	MSE	$\hat{\alpha}$	$\hat{\beta}$	MSE	$\hat{\alpha}$	$\hat{\beta}$	MSE
10	0.7181	3.2216	3.9120	0.9913	3.9211	4.3721	0.7980	3.5531	4.0055
20	0.6971	3.2006	3.8910	0.9703	3.9001	4.3511	0.7701	3.5241	3.9765
30	0.6771	3.1806	3.8711	0.9503	3.8801	4.3311	0.7431	3.4961	3.9485
40	0.6581	3.1616	3.8521	0.9313	3.8611	4.3121	0.7171	3.4691	3.9205
50	0.6401	3.1436	3.8341	0.9133	3.8431	4.2941	0.6911	3.4431	3.8935
60	0.6231	3.1266	3.8171	0.8963	3.8261	4.2771	0.6661	3.4171	3.8665
70	0.6071	3.1096	3.8011	0.8813	3.8101	4.2611	0.6411	3.3921	3.8405
80	0.5921	3.0936	3.7861	0.8673	3.7951	4.2461	0.6171	3.3681	3.8145
90	0.5781	3.0776	3.7731	0.8543	3.7811	4.2311	0.5941	3.3451	3.7885
100	0.5651	3.0626	3.7611	0.8423	3.7671	4.2181	0.5721	3.3231	3.7635
120	0.5531	3.0486	3.7501	0.8313	3.7541	4.2051	0.5511	3.3011	3.7385
140	0.5411	3.0356	3.7401	0.8203	3.7421	4.1931	0.5471	3.2801	3.7145
160	0.5301	3.0236	3.7311	0.8103	3.7301	4.1811	0.5321	3.2601	3.6905
180	0.5191	3.0126	3.7231	0.8013	3.7191	4.1701	0.5110	3.2411	3.6665
200	0.5101	3.0026	3.7161	0.7933	3.7091	4.1591	0.5001	3.2231	3.6425

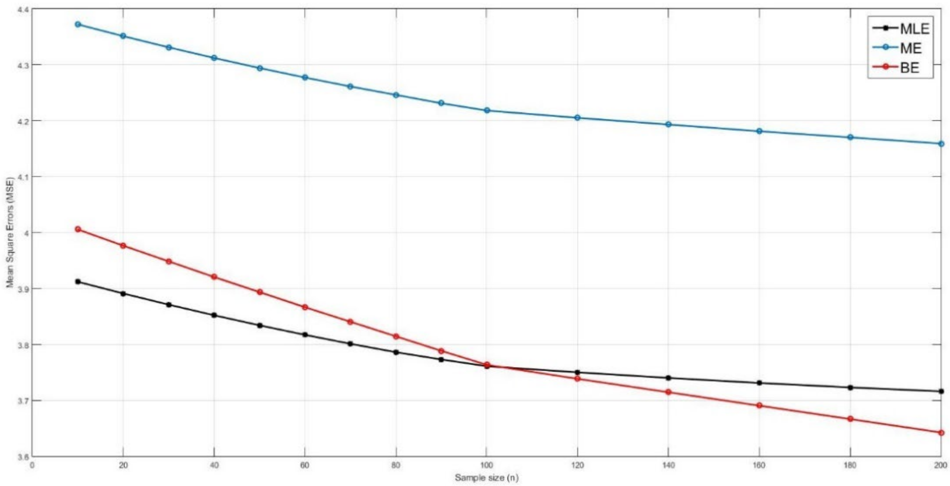

Figure 3: The values of MSE for the three estimation methods: Maximum Likelihood Estimation Method, Moment Estimation and Bayes Estimation, when $\boldsymbol{\alpha} = \mathbf{0.5}$, $\boldsymbol{\beta} = \mathbf{2.5}$

By changing $\boldsymbol{\beta} = \mathbf{2.5}$ Table 3 and Figure 3 show no change except that the BE estimator is better than the MLE estimator when the sample size is 120 and MLE is better when the sample size is smaller than 120, while the ME estimator is the same as above.

8. Discussion

We believe that estimating the parameters of the Weibull Distribution function is an essential procedure in many statistical applications. In Section 7, we made a simulation by fixed shape parameter and many scale parameter values to cover multiple cases of this distribution function by increasing the scale parameter, $\boldsymbol{\alpha}$ will increase the peak of the probability density function. From Figures 1, 2 and 3, we can see that the Bayes Estimator will have better performance as it increases in scale parameter $\boldsymbol{\alpha}$ and sample size either there is good performance of the Maximum likelihood Estimator MLE when both scale parameter $\boldsymbol{\alpha}$ and the sample size are small. By increasing the scale parameter $\boldsymbol{\alpha}$, we can see the MLE preference decreases to smaller sample sizes. As for the Moment Estimator ME, we can see from all figures that changing the scale parameter $\boldsymbol{\alpha}$ did not result in a good quality estimator in all sample sizes.

9. Conclusions

This paper compares and demonstrates three methods for estimating parameters: Maximum Likelihood Estimator, Moment Estimator, and Bayesian Estimator with a non-informative prior (weak prior with minimal influence).

The paper uses Monte Carlo simulations and analyzes results from tables and figures (not included here).

The Maximum Likelihood Estimator outperforms the Moment Estimator and Bayesian Estimator for small sample sizes. Concerning larger sample sizes: Maximum Likelihood Estimator and Bayesian Estimator perform better than Moment Estimator.

Finally, for large sample sizes, Bayesian Estimator becomes significantly better than the Maximum Likelihood Estimator and Moment Estimator for parameter estimation.

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